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Equilibrium Configurations of a Four-Body Tethered System

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I. Introduction

SINCE the early 1970s, tethered satellite systems have been considered and studied for space missions, providing a number of useful applications [1]: creation of artificial gravity, generation of thrust maneuvers and exchange of angular momentum, atmospheric studies, etc. The key characteristic that makes the use of tethers appealing is lightness. In space, the forces needed to keep objects together using a tether are small, thus very thin cables can be used to connect satellites, and small sections mean small weights, an essential requirement for space operations.

In 2002, Misra [2] performed an analytical study of the planar three-body tethered system, including the linear stability of their equilibrium configurations. He concluded that the triangular configurations of the system are unstable whereas one of the collinear configurations is stable. Following the analytical formulation given in Misra's work, Tan and Bainum [3] have considered nonrigid tethered systems using a three-body configuration. The authors suggested a tetrahedron tethered system for Earth's aurora observation missions and gave a preliminary design of a controller for use with an orbiting tethered system in formation flying.

The present paper studies the equilibrium configurations of a four-body tethered system. The study is based on the analytical development done by Misra [2] for the three-body system. Once the equilibrium solutions of this last problem are known, then the basic idea is to continue these solutions when one of the bodies of the system splits into two pieces. The continuation procedure introduced can be extended to a n-body system.

II. Equations of Motion

Consider a system of four point masses m_1, m_2, m_3, and m_4, connected by tethers of lengths l_1, l_2, and l_3 (with l_i joining m_i and m_{i+1}) and of total mass m. The tethers are assumed inextensible and massless. The four bodies are also assumed to move on a plane in such way that the baricenter of the system moves along a circular orbit around the Earth with angular velocity Ω. As shown in Fig. 1, a reference frame is introduced that has its origin at the baricenter of the system; the x-axis is along the Earth—baricenter line, and the y-axis is parallel to the baricenter's velocity vector.

To derive the equations of motion for the tethered system, expressions for the kinetic and potential energies of the system are generated, and then substituted into Lagrange's equation. The resulting equations of motion are

$$\begin{aligned}&\mu_1(1-\mu_1)l_1^2(\ddot\theta_1+3\Omega^2\cos\theta_1\sin\theta_1)\\&\quad+\mu_1(\mu_3+\mu_4)l_1l_2[\ddot\theta_2\cos(\theta_1-\theta_2)+3\Omega^2\cos\theta_2\sin\theta_1]\\&\quad+\mu_1\mu_4l_1l_3[\ddot\theta_3\cos(\theta_1-\theta_3)+3\Omega^2\cos\theta_3\sin\theta_1]\\&\quad+\mu_1(\mu_3+\mu_4)l_1l_2(\dot\theta_2^2+2\Omega\dot\theta_2)\sin(\theta_1-\theta_2)\\&\quad+\mu_1\mu_4l_1l_3(\dot\theta_3^2+2\Omega\dot\theta_3)\sin(\theta_1-\theta_3)=0\end{aligned}\tag{1}$$

$$\begin{aligned}&(\mu_1+\mu_2)(\mu_3+\mu_4)l_2^2(\ddot\theta_2+3\Omega^2\cos\theta_2\sin\theta_2)\\&\quad+\mu_1(\mu_3+\mu_4)l_1l_2[\ddot\theta_1\cos(\theta_1-\theta_2)+3\Omega^2\cos\theta_1\sin\theta_2]\\&\quad+\mu_4(\mu_1+\mu_2)l_2l_3[\ddot\theta_3\cos(\theta_2-\theta_3)+3\Omega^2\cos\theta_3\sin\theta_2]\\&\quad+\mu_1(\mu_3+\mu_4)l_1l_2(\dot\theta_1^2+2\Omega\dot\theta_1)\sin(\theta_2-\theta_1)\\&\quad+\mu_4(\mu_1+\mu_2)l_2l_3(\dot\theta_3^2+2\Omega\dot\theta_3)\sin(\theta_2-\theta_3)=0\end{aligned}\tag{2}$$

$$\begin{aligned}&\mu_4(1-\mu_4)l_3^2(\ddot\theta_3+\Omega^2\cos\theta_3\sin\theta_3)\\&\quad+\mu_4(\mu_1+\mu_2)l_2l_3[\ddot\theta_2\cos(\theta_2-\theta_3)+3\Omega^2\cos\theta_2\sin\theta_3]\\&\quad+\mu_1\mu_4l_1l_3[\ddot\theta_1\cos(\theta_1-\theta_3)+3\Omega^2\cos\theta_1\sin\theta_3]\\&\quad+\mu_1\mu_4l_1l_3(\dot\theta_1^2+2\Omega\dot\theta_1)\sin(\theta_3-\theta_1)\\&\quad+\mu_4(\mu_1+\mu_2)l_2l_3(\dot\theta_2^2+2\Omega\dot\theta_2)\sin(\theta_3-\theta_2)=0\end{aligned}\tag{3}$$

where $\mu_i=m_i/m$ so that $\sum_{i=1}^4\mu_i=1$. The generalized coordinate θ_i is the angle between the tether and the x-axis.

III. Analytical Continuation of the Equilibrium Solutions

Let f_1, f_2, f_3 be the analytical functions that remain when the derivative terms of θ_i are set to zero in Eqs. (1–3)

$$f_i(\theta_1,\theta_2,\theta_3)=\sin\theta_i\sum_{j=1}^3 b_{ij}\cos\theta_j,\qquad i=1,2,3\tag{4}$$

where b_{ij} are the components of the matrix

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$$B = 3\Omega^2 \begin{bmatrix} \mu_1(1-\mu_1)l_1^2 & \mu_1(\mu_3+\mu_4)l_1l_2 & \mu_1\mu_4l_1l_3 \\ \mu_1(\mu_3+\mu_4)l_1l_2 & (\mu_1+\mu_2)(\mu_3+\mu_4)l_2^2 & \mu_4(\mu_1+\mu_2)l_2l_3 \\ \mu_1\mu_4l_1l_3 & \mu_4(\mu_1+\mu_2)l_2l_3 & \mu_4(1-\mu_4)l_3^2 \end{bmatrix}$$

The solutions of

$$\mathbf{F}(\theta_1, \theta_2, \theta_3) = (f_1, f_2, f_3) = 0 \tag{5}$$

give the equilibrium configurations of the tethered system.

The baricentric coordinates, (μ_1, μ_2, μ_3), $\sum_{i=1}^{3}\mu_i = 1$, of any point inside the triangle of mass represents a certain mass distribution for the three bodies of the system (see Fig. 2). If the body of mass μ_3 is divided in two pieces, $\mu_3 \to (\mu_3, \mu_4)$ with $\sum_{i=1}^{4}\mu_i = 1$, then the triangle must be replaced by the tetrahedron of masses. All the possible values of the masses of the bodies are represented by the points on and in the tetrahedron. If, for instance, we consider mass distributions with $\mu_1 = \mu_2$, then the admissible values of the masses are those represented by the shaded triangle of Fig. 2. Because we are going to use a continuation procedure, starting with the equilibrium configurations of the three-body system, we will follow a certain path in this tetrahedron, starting on its base ($\mu_4 = 0$). To this end, we introduce a continuation parameter ϵ, so that Eq. (5) becomes

$$\mathbf{G}(\theta_1, \theta_2, \theta_3; \epsilon) = 0 \tag{6}$$

Now, if $\epsilon = 0$ the solutions of Eq. (6) must give the three-body equilibrium configurations and if $\epsilon > 0$, we obtain those of the four-body case. To get the final mass values $(\mu_1, \mu_2, \mu_3, \mu_4) = (\mu, \mu, \mu, 2\mu)$ and the final length of the tethers $l_1 = l_2 = l_3 = 1$, one possible choice for the continuation procedure is

$$\mu_3 = (1-\epsilon)\mu_3^0 \quad \text{and} \quad \mu_4 = \epsilon\mu_3^0, \qquad l_3 = \frac{3}{2}\epsilon l_2$$

where $\mu_3^0 = 0.6$ is the starting value for μ_3^0. The continuation process will always start at the initial equilibrium configurations obtained for $(\mu_1, \mu_2, \mu_3) = (\mu, \mu, 3\mu)$ with $\mu = 0.2$.

IV. Linear Stability

The linear variational equations are obtained by introducing small displacements $\delta\theta_i$ in the equilibrium configurations under consideration

$$\theta_i = \theta_i^0 + \delta\theta_i, \quad \text{with } i = 1\ldots3 \tag{7}$$

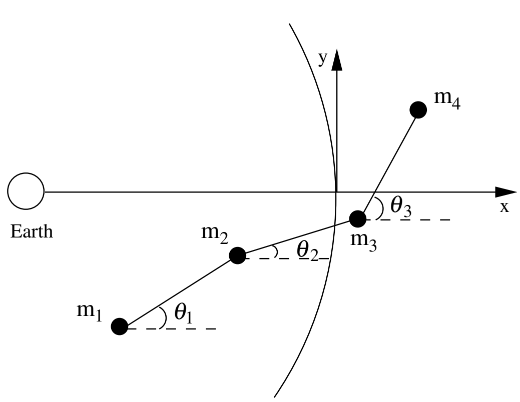

Fig. 1 Reference system and angular coordinates for the planar four-body tethered system.

The linearization of the differential equations for $\delta\theta_i$, after introducing a new nondimensional independent variable τ with $\tau = \Omega t$, and denoting the derivatives with respect to τ with a prime, gives

$$M\delta\Theta'' + C\delta\Theta' + K\delta\Theta = 0 \tag{8}$$

where the matrices M, C, and K depend on the parameter ϵ. The solutions of Eq. (8) are of type

$$\delta\Theta = \delta\Theta^0 \exp(\lambda\tau) \tag{9}$$

so that the linear stability behavior is determined by the characteristic exponents λ [4]. The equilibrium solutions are asymptotically stable if all the exponents λ have negative real part; if there is at least one λ with a positive real part, then the equilibrium solutions are unstable. If the linear system has no root with positive real part but has some imaginary or null root, then the stability analysis will depend on the full system and not only on its linearization (marginal stable solutions). Because the system of differential equations can be written in Hamiltonian form and Hamiltonian systems do not have asymptotically stable/unstable solutions, the equilibrium configurations of the tethered system do not have the asymptotic behavior. The numerical routines available in Press et al. [5] were used to find the numerical values λ.

V. Equilibrium Configurations

A. Trivial Equilibrium Configurations

The most trivial solutions of Eq. (4) are those which cancel the $\sin\theta_i$ terms of these equations. They are obtained by setting $\theta_i^0 = 0, \pi$. In this way, we get the following eight equilibrium configurations:

$$S_{1a} = \{(0,0,0),(0,0,\pi)\}, \qquad S_{1b} = \{(\pi,0,0),(\pi,0,\pi)\}$$
$$S_{1c} = \{(0,\pi,0),(0,\pi,\pi)\}, \qquad S_{1d} = \{(\pi,\pi,0),(\pi,\pi,\pi)\}$$

All the preceding eight configurations are vertically aligned and are displayed in Fig. 3. These solutions could also be obtained using the continuation procedure already explained, taking as initial states the E_{1a}, E_{1b}, E_{1c}, and E_{1d} configurations classified by Misra [2].

A second set of trivial solutions is obtained when the $\cos\theta_i$ terms of Eq. (4) vanish. The different possibilities that we have in this situation give the following equilibrium configurations:

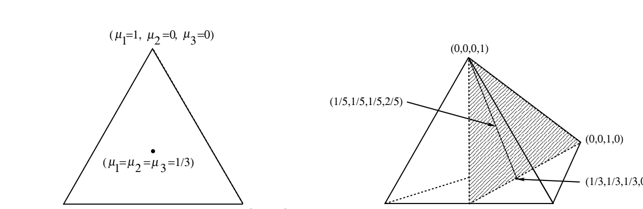

Fig. 2 The triangle and tetrahedron of masses.

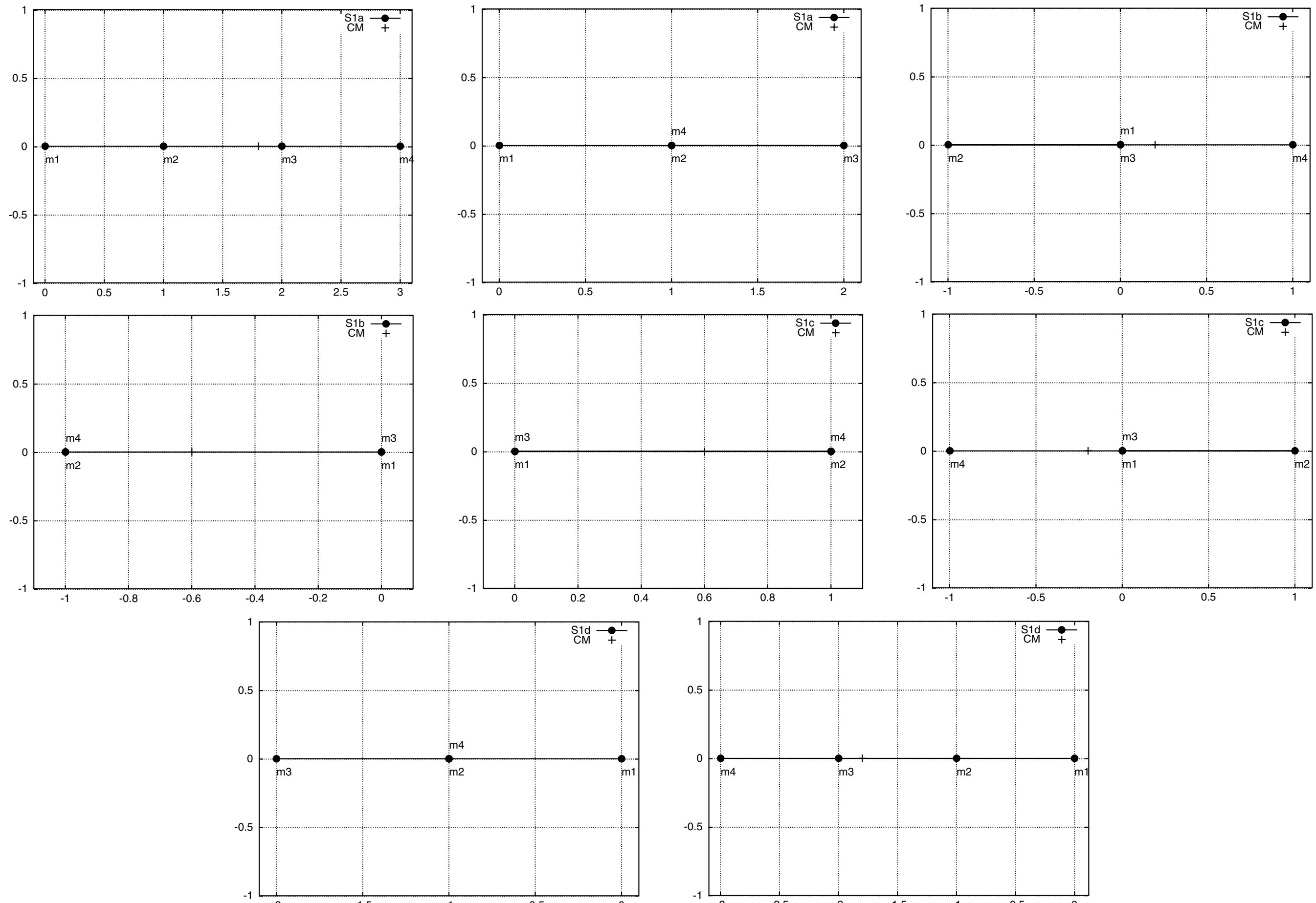

Fig. 3 Trivial equilibrium configurations: S_{1a} and S_{1d} are stable; S_{1b} and S_{1c} are unstable.

$$S_{2a} = \{(\pi/2, \pi/2, \pi/2), (\pi/2, \pi/2, -\pi/2)\}$$
$$S_{2b} = \{(-\pi/2, \pi/2, \pi/2), (-\pi/2, \pi/2, -\pi/2)\}$$
$$S_{2c} = \{(-\pi/2, -\pi/2, \pi/2), (-\pi/2, -\pi/2, -\pi/2)\}$$
$$S_{2d} = \{(\pi/2, -\pi/2, \pi/2), (\pi/2, -\pi/2, -\pi/2)\}$$

These solutions, that are shown in Fig. 4, are all horizontally aligned and can also be obtained using the continuation procedure, taking as initial states the configurations E_{2a}, E_{2b}, E_{2c}, and E_{2d} of Misra [2].

All the S_2 configurations are unstable, as in the case of three-body tethered system, and the vertically aligned configurations S_{1a} and S_{1d} are linearly stable whereas the S_{1b} and S_{1c} are unstable.

B. The Nontrivial Solutions

The nontrivial solutions of the system formed by Eq. (4) are found by simple Gaussian elimination. There are two main equilibrium groups: in the first group, only one of the three equilibrium angles is fixed, and in the second one, the values of two angles are fixed. In the first group we have not taken into account the symmetric cases, by fixing $\theta_i = \pi$. These equilibrium solutions are obtained from the continuation procedure through the parameter ϵ. They are shown in Tables 1 and 2 and displayed in Figs. 5 and 6.

The equilibrium configurations S_3 can be seen as natural extensions of E_{3a} and E_{3b}, because the cosine argument does not change for all ϵ. For the S_4 case, it is clear that if $\epsilon = 0$ the angle θ_3 is not defined and the denominator is smaller than the numerator for $\epsilon \leq 0.195263$. For this reason, the idea of continuation of the solutions does not make sense (at least for values of ϵ smaller than 0.195263). The equilibrium solutions of S_5 group have been continued from those of the three-body configurations E_2. All these configurations are unstable.

The equilibrium configurations of S_6 have been continued from the three-body configurations E_{4a} and E_{4b}. The S_7 subset were born from E_{3a} of the three-body tethered system. When the angle $\theta_2 =$ 120 deg increases towards 180 deg the solution does not exist because the cosine argument is greater than one; this happens if $\epsilon \geq 0.57735$. These configurations are unstable. For the S_8 group, the unique solution of θ_3 occurs only for $\epsilon = 0.666667$, meaning that no continuation procedure could be applied; these configurations are equal to the ones in S_1 which are stable. The other two solutions of S_8 are unstable.

VI. Conclusions

Equilibrium configurations of a four-body tethered systems have been found through the use of a parameterization that takes the three-body tethered configuration to the four-body one. The equilibrium solutions are given in terms of trigonometric functions and it has been shown that some solutions do not exist for any point in the tetrahedron of masses $\triangle(\mu_1 = \mu_2, \mu_3, \mu_4)$ during the continuation procedure. When the third body splits into two bodies, two possibilities for the four-body equilibrium solutions arise, thus there are 16 trivial equilibrium solutions and 24 nontrivial equilibrium solutions. However, the stability properties of the three-body solutions can vary along the continuation up to the four-body problem.

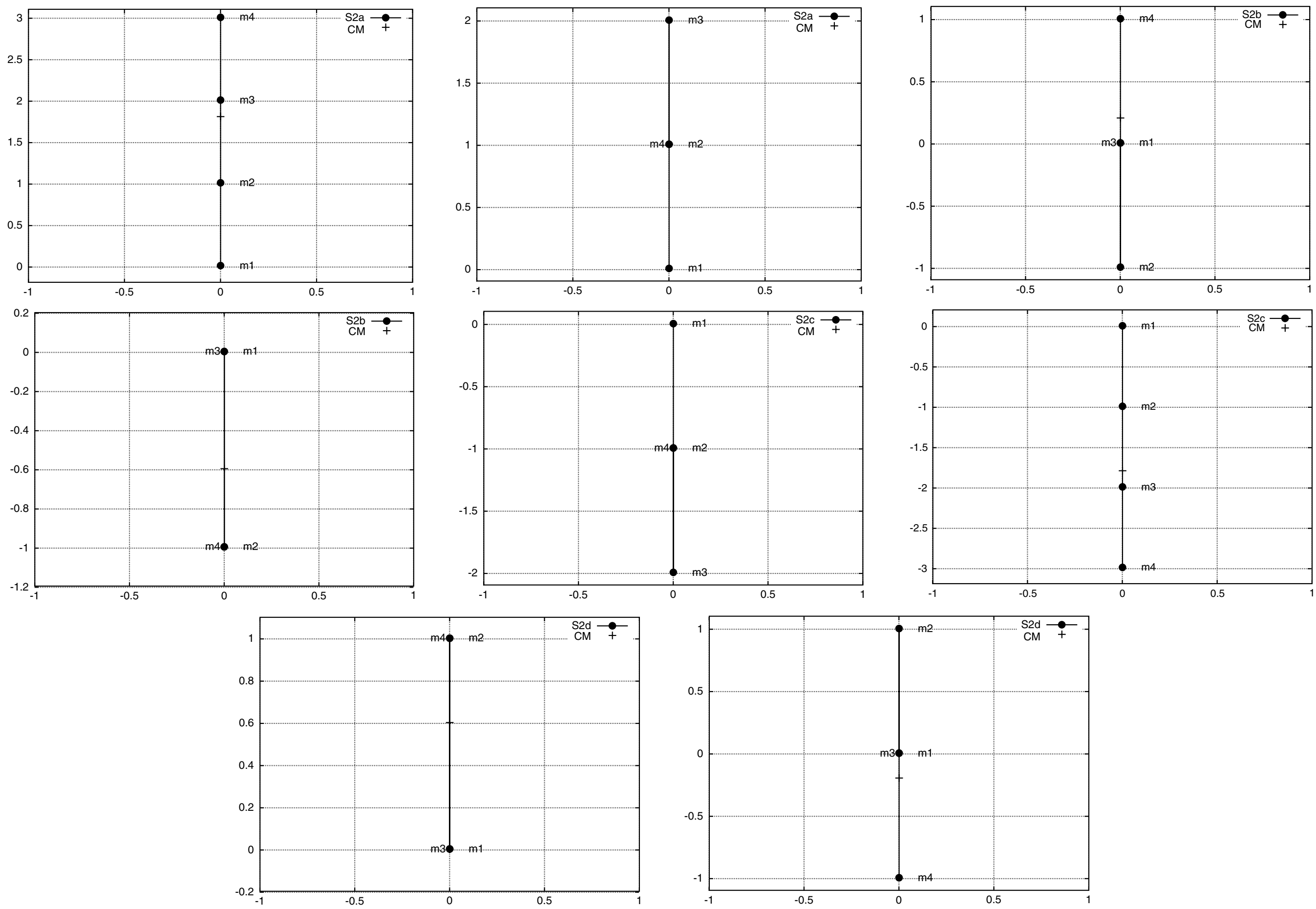

Fig. 4 The S_{2a}, S_{2b}, S_{2c}, and S_{2d} trivial configurations. They are all unstable.

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Table 1 Equilibrium solutions of the nontrivial cases

Group	θ_1	θ_2	θ_3	Range of ϵ
S_3	0	$\pm\cos^{-1}\left(-\frac{\mu_1 l_1}{(\mu_1+\mu_2)l_2}\right)$	$\pm\frac{\pi}{2}$	(0.0, 0.666667)
S_4	$\pm\cos^{-1}\left(\frac{(1-\epsilon)\mu_3 l_2}{(\epsilon\mu_3+\mu_1-1)l_1}\right)$	0	$\pm\cos^{-1}\left(\frac{2}{3}\frac{\mu_2 l_2}{\epsilon(\epsilon\mu_3+\mu_1-1)l_1}\right)$	(0.195263, 0.666667)
S_5	$\pm\frac{\pi}{2}$	$\pm\cos^{-1}\left(-\frac{3}{2}\frac{\epsilon^2 l_1}{l_2}\right)$	0	(0.0, 0.666667)

Table 2 Equilibrium solutions of the nontrivial cases

Group			Range of ϵ
S_6	$\theta_1=\pm\cos^{-1}\left(\frac{2\mu_3 l_2+3\epsilon^2\mu_3 l_1}{2(1-\mu_1)l_1}\right)$	$(\theta_2,\theta_3)=(0,0)$ or (π,π)	(0.0, 0.471404)
	$\theta_1=\pm\cos^{-1}\left(\frac{2\mu_3 l_2-3\epsilon^2\mu_3 l_1}{2(1-\mu_1)l_1}\right)$	$(\theta_2,\theta_3)=(0,\pi)$ or $(\pi,0)$	(0.0, 0.666667)
S_7	$(\theta_1,\theta_3)=(0,0)$ or (π,π)	$\theta_2=\pm\cos^{-1}\left(-\frac{3\epsilon^2(\mu_1+\mu_2)l_1+2\mu_1 l_1}{2(\mu_1+\mu_2)l_2}\right)$	(0.0, 0.57735)
	$(\theta_1,\theta_3)=(0,\pi)$ or $(\pi,0)$	$\theta_2=\pm\cos^{-1}\left(-\frac{3\epsilon^2(\mu_1+\mu_2)l_1-2\mu_1 l_1}{2(\mu_1+\mu_2)l_2}\right)$	(0.0, 0.666667)
S_8	$(\theta_1,\theta_2)=(0,0)$ or (π,π)	$\theta_3=\pm\cos^{-1}\left(-\frac{2(\mu_1+\mu_2)l_2+2\mu_1 l_1}{3\epsilon(1-\epsilon\mu_3)l_1}\right)$	only for $\epsilon=0, 0.666667$
	$(\theta_1,\theta_2)=(0,\pi)$ or $(\pi,0)$	$\theta_3=\pm\cos^{-1}\left(-\frac{2(\mu_1\mu_2)l_2-2\mu_1 l_1}{3\epsilon(1-\epsilon\mu_3)l_1}\right)$	(0.14615, 0.666667)

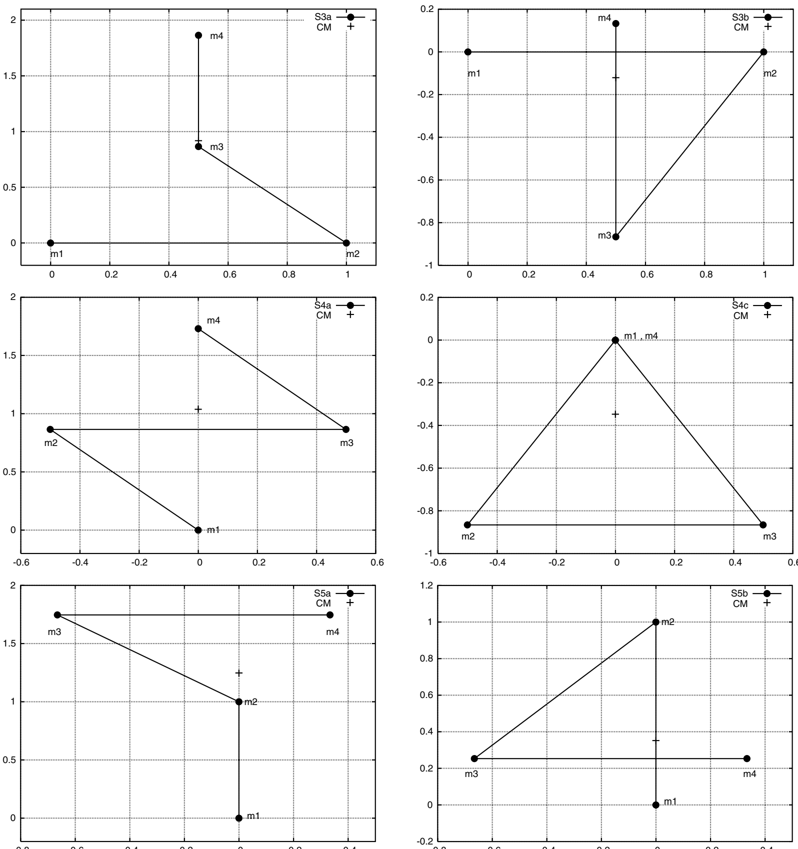

Fig. 5 From top to bottom, each pair of equilibrium configurations corresponds to S_3,S_4, and S_5. The symmetric cases are not displayed.

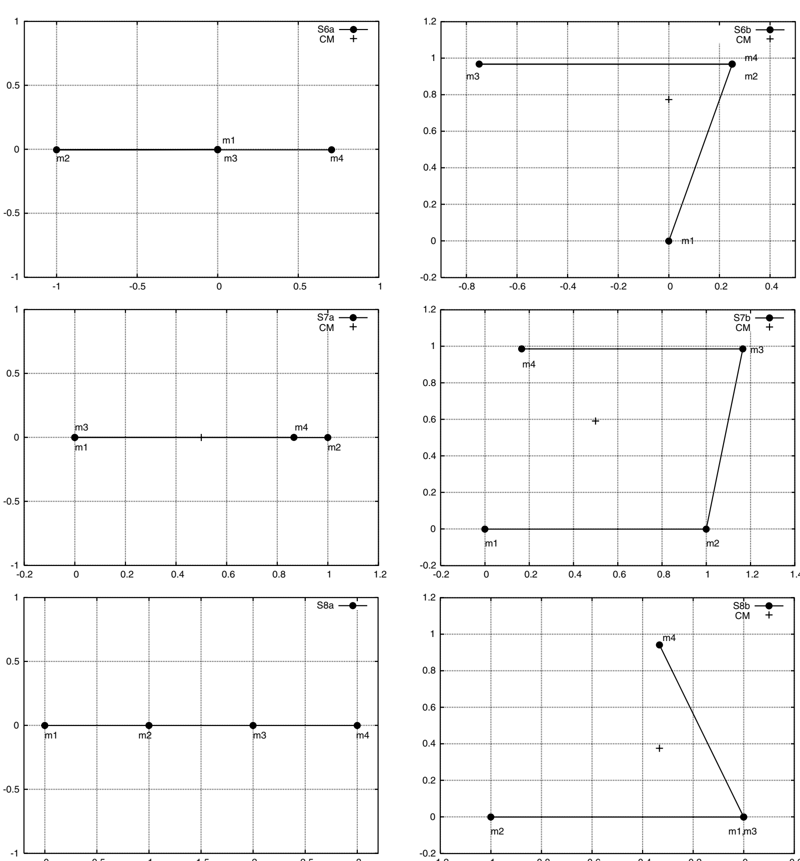

Fig. 6 From top to bottom, each pair of equilibrium configurations corresponds to S_6,S_7, and S_8. The symmetric cases are not displayed.

References

[1] Cosmo, M. L., and Lorenzini, E. C., *Tethers in Space Handbook*, NASA Marshall Space Flight Center, Huntsville, AL, 1997.

[2] Misra, A. K., "Equilibrium Configurations of Tethered Three-Body Systems and Their Stability," *The Journal of the Astronautical Sciences*, Vol. 50, No. 3, 2002, pp. 241–253.

[3] Tan, Z., and Bainum, P. M., "Tethered Satellite Constellations in Auroral Observations Missions," AIAA Paper 2002-4640, Aug. 2002.

[4] Meirovitch, L., *Introduction to Dynamics and Control*, Wiley, New York, 1985, Chap. 9.

[5] Press, W. H., Teukolsky, S. A., Vettering, W. T., and Flannery, B. P., *Numerical Recipes in Fortran*, 2nd ed., Cambridge Univ. Press, Cambridge, England, U.K., 1993, pp. 362–372.